Problema (2)

$$f_a(x) = \frac{x^2 - ax}{x^2 - a} = \frac{x(x-a)}{(x+\sqrt{a})(x-\sqrt{a})} \qquad a \neq 0$$

$D[f_a(x)] = \mathbb{R}$ se $a < 0$

$D[f_a(x)] = (-\infty, -\sqrt{a}) \cup [-\sqrt{a}, \sqrt{a}] \cup (\sqrt{a}, +\infty)$ se $a > 0$

Per $a < 0$ non ci sono discontinuità

$a > 0$ $x = \pm\sqrt{a}$ sono pt di discontinuità

$\lim_{x \to \pm\infty} f_a(x) = 1 \quad \forall a \neq 0$, $y = 1$ asintoto orizzon.

$\lim_{x \to \sqrt{a}^+} f_a(x) = -\infty$

$\lim_{x \to \sqrt{a}^-} f_a(x) = +\infty$, $x = \sqrt{a}$ asint. verticale

$\lim_{x \to -\sqrt{a}^+} f_a(x) = -\infty$

$\lim_{x \to -\sqrt{a}^-} f_a(x) = +\infty$, $x = -\sqrt{a}$ asint. verticale

b) $$\frac{df(x)}{dx} = \frac{(2x-a)(x^2-a) - (x^2-ax)(2x)}{(x^2-a)^2} =$$

$$= \frac{\cancel{2x^3} - 2ax - ax^2 + a^2 - \cancel{2x^3} + 2ax^2}{(x^2-a)^2} =$$

$$= \frac{a(a + x^2 - 2x)}{(x^2-a)^2} = \frac{a(a + x(x-2))}{(x^2-a)^2}$$

$$\left.\frac{df(x)}{dx}\right|_{x=0} = 1 \qquad \forall\, a$$

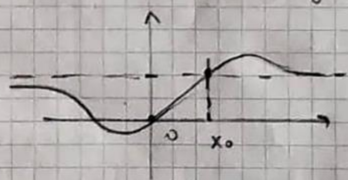
$y(x) = f(0) + f'(0)(x-0) = x \quad \forall\, a$, retta tangente

$f'(x) = 0$ in $a + x(x-2) = 0$

$$x^2 - 2x + a = 0 \longrightarrow x_{\pm} = \frac{2 \pm \sqrt{4-4a}}{2} = 1 \pm \sqrt{1-a}$$

Vale per $a < 1$ ... $\exists$ pt stazionari

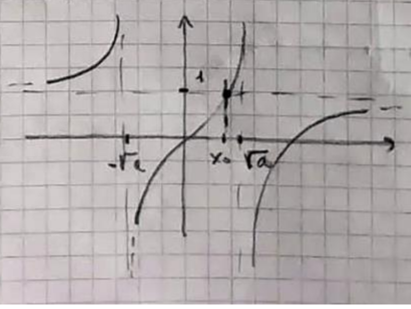
$$\begin{cases} f(x_+) > 1 \\ f(x_-) < 1 \end{cases} \longrightarrow \exists\, x_0 \text{ tale che } f(x_0) = 1$$



$x = 1$

per $a < 0$

Per $a > 0$ basta notare che $f_a(x) \longrightarrow \pm\infty$ in $x = \pm\sqrt{a}$ e quindi $\exists\, x_0$ tale che $f(x_0) = 1$



$x = 1$

c) Per $a < 1$ voglio studiare la monotonia

$f'(x) = 0$ in $x_{\pm} = 1 \pm \sqrt{1-a}$

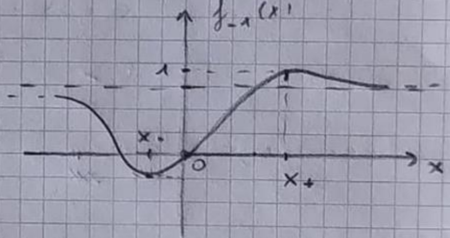
Per
$$\begin{cases} x < x_- & f(x) \text{ monotona decrescente} \\ x_- < x < x_+ & f(x) \text{ monotona crescente} \\ x > x_+ & f(x) \text{ monotona decrescente} \end{cases}$$

infatti $f'(x) = (x - x_+)(x - x_-) > 0$ in $x_- < x < x_+$

$f'(x) < 0$ in $x < x_-$ o $x > x_+$

Studio la funzione per $a = -1$

$$f_{-1}(x) = \frac{x(x+1)}{x^2+1}$$

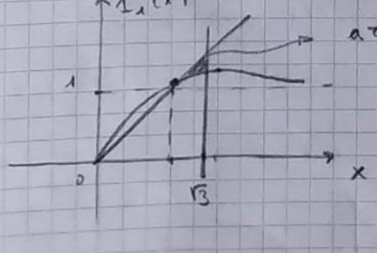


$f'_{-1}(x) = 0$ in $x_{\pm} = 1 \pm \sqrt{2}$

$$f_{-1}(x_+) = \frac{(1+\sqrt{2})(2+\sqrt{2})}{(1+\sqrt{2})^2+1} \simeq 1,2$$

$f_{-1}(x_-) \simeq -0,2$

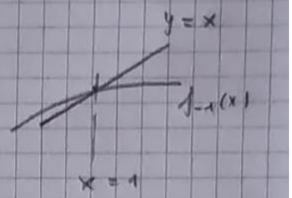
d) Calcolo l'area compresa



$$f_{-1}(x) = \frac{x(x+1)}{x^2+1} = x \quad \text{se } x = 0 \text{ o } x = 1$$

In generale $f_{-1}(x) > x$ per $x < 1$

$< x$ per $x > 1$



Voglio calcolare l'integrale

$$\int_1^{\sqrt{3}} \left(x - f_{-1}(x)\right) dx = \int_1^{\sqrt{3}} \left(x - \frac{x(x+1)}{x^2+1}\right) dx =$$

$$= \int_1^{\sqrt{3}} x \, dx - \int_1^{\sqrt{3}} \left(\frac{x^2+x}{x^2+1}\right) dx =$$

$$= \frac{x^2}{2}\Big|_1^{\sqrt{3}} - \int_1^{\sqrt{3}} \left[\frac{(x^2+2x+1)-(x-1)}{x^2+1}\right] dx =$$

$$= 1 - \int_1^{\sqrt{3}} \left(1 - \left[\frac{(x-1)}{x^2+1}\right]\right) dx =$$

$$= 1 - \sqrt{3} + 1 + \int_1^{\sqrt{3}} \left(\frac{x-1}{x^2+1}\right) dx =$$

$$= 2 - \sqrt{3} + \int_1^{\sqrt{3}} \frac{x}{x^2+1} dx - \int_1^{\sqrt{3}} \frac{1}{x^2+1} dx =$$

$$= 2 - \sqrt{3} + \frac{1}{2} \ln(x^2+1)\Big|_1^{\sqrt{3}} - \arctan x \Big|_1^{\sqrt{3}} =$$

$$= 2 - \sqrt{3} + \frac{\ln\left(\frac{\sqrt{3}+1}{2}\right)}{2} - \arctan\sqrt{3} + \arctan 1 \cong 0,41$$

Fonte: Studenti.it - https://www.studenti.it/problema-2-svolto-della-traccia-di-matematica-seconda-prova-2023.html